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This article presents the derivation of the stress-energy tensor of a free scalar field with a general non-linear dispersion relation in curved spacetime. This dispersion relation is used as a phenomenological description of the short distance structure of spacetime following the conventional approach of trans-Planckian modes in black hole physics and in cosmology. This stress-energy tensor is then used to discuss both the equation of state of trans-Planckian modes in cosmology and the magnitude of their backreaction during inflation. It is shown that gravitational waves of trans-Planckian momenta but subhorizon frequencies cannot account for the form of cosmic vacuum energy density observed at present, contrary to a recent claim. The backreaction effects during inflation are confirmed to be important and generic for those dispersion relations that are liable to induce changes in the power spectrum of metric fluctuations. Finally, it is shown that in pure de Sitter inflation there is no modification of the power spectrum independently of the dispersion relation.

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I. INTRODUCTION

The inflationary paradigm provides an appealing framework to describe the very early phase of the evolution of the Universe, notably because it produces in a natural way the seeds necessary to the formation of the cosmological large scale structures [1]. These initial density fluctuations are generated during inflation with an almost scale invariant power spectrum through the parametric amplification of the quantum fluctuations of the inflaton scalar field, with the accelerated Friedman-Lemaître scale factor acting as a classical pump field. This can also be seen as particle production due to the breakdown of adiabaticity in the evolution of the quantum modes of the field as the wavelengths are stretched beyond the horizon.

In spite of their successes, most models of inflation are subject to the *trans-Planckian problem*, namely the phase of accelerated expansion lasts sufficiently long that cosmological length scales today corresponded to scales much smaller than the Planck scale at the beginning of inflation [2,3]. Depending on one's point of view, this can be seen either as a problem, i.e. the celebrated predictions depend on unknown trans-Planckian physics, or as a blessing, since inflation then opens a window on physics beyond the Planck scale. Several studies have recently tackled the issue of the robustness of inflation to changes in super-Planck physics by adopting a phenomenological approach initially developed in the context of black hole physics, where a very similar problem arises [4,5], and in close relation to analogous problems in condensed matter physics [6]. It consists in modifying the standard dispersion relation of a free scalar field for wavelengths smaller than the Planck length and in calculating the resulting power spectrum of inflation produced metric fluctuations [2,3,7,8]. This approach is further motivated by the fact that the evolution of the scalar and tensor modes of metric fluctuations can be adequately described by free scalar fields propagating in a background spacetime [9]. It should be noted that this is only a phenomenological approach to the problem of trans-Planckian physics, and that it is not unique. For instance, modifications of the canonical quantum operator commutation relations inspired from string theories [10] have also been shown to affect the power spectrum of metric fluctuations [11].

Interestingly, concrete examples of dispersion relations leading to modifications of the power spectrum with respect to the standard (linear dispersion relation) predictions have been exhibited [2,3] but only in the case where the dispersion relation becomes complex. Even though complex dispersion relations are ordinary in classical physics or in quantum mechanics, they represent a problematic situation in the context of quantum field theory. It has also been shown that dispersion relations such that the evolution of the quantum mode is adiabatic all throughout the inflationary phase up to horizon crossing cannot lead to significant modifications of the power spectrum [12]. With

respect to the above example of dispersion relations, indeed adiabaticity is broken at the point where the physical frequency vanishes and the dispersion relation becomes complex. However it has been argued that such dispersion relations which break adiabaticity would lead to a possibly severe backreaction problem, i.e. the energy density contained in the modes would become greater than the background energy density [13,14]. Therefore modifications to the power spectrum may arise if adiabaticity is broken, but at the expense of the creation of a possibly large amount of energy density. Unfortunately this amount could not be quantified rigorously as the authors did not have at their disposal the stress-energy tensor of a theory with modified dispersion relation.

In this context, our main objective is to present a rigorous derivation of the stress-energy tensor of a free scalar field with non-linear dispersion relation. Such a dispersion relation breaks local Lorentz invariance as it implies the existence of a preferred reference frame, however it is crucial to maintain general covariance in order to achieve consistency with the Einstein equations (and notably the conservation of the stress-energy tensor). This problem has been examined in a number of studies, where an effective general covariant Lagrangian with explicit Lorentz invariance breaking has been constructed with the introduction of a dynamical unit timelike vector field whose role is to define the preferred rest frame [15,16]. An effective Lagrangian describing a free scalar theory with a quartic dispersion relation could then be constructed along these lines, and the stress-energy tensor has been calculated for the particular case of the Jacobson-Corley dispersion relation [16,17]. In the present paper we present a non-trivial extension of this latter study to the general case in which the squared frequency is a general analytic function of the squared momentum. Note that in Friedman-Lemaître-Robertson-Walker (FLRW) cosmology, a preferred rest frame exists and coincides with the homogeneous isotropic spatial sections.

We then use this stress-energy tensor to address two points raised recently in the literature. We first discuss the claim [18] according to which a bath of gravitons of super-Planck momenta but frequencies much smaller than the Hubble expansion rate (hence a particular non-linear dispersion relation), can explain the form of vacuum energy seen in the Universe today. In order to do so, we calculate the energy density, the pressure and the equation of state of these quanta; we show that these gravitons possess neither the correct energy density nor the correct equation of state. As a second application we discuss the issue of backreaction of trans-Planckian modes in inflationary cosmology, using the expression for the energy density contained in trans-Planckian modes.

This article is organized as follows. In Section II we formulate a covariant Lagrangian (§ II A) including extra terms to provide a modified dispersion relations and calculate the corresponding stress-energy tensor (§ II B). Then, we specify our result to the case of cosmology (§ II C). Section III discusses the two main applications of our results, namely the equation of state of the trans-Planckian modes (§ III A) and the backreaction problem (§ III B and § III C). Our results are summarized in Section IV. For the sake of clarity, we gathered notations and derivations of various identities used in the calculation of the stress-energy tensor in Appendix A; Appendix B presents the detailed derivation of the stress-energy tensor in the simpler case of the Jacobson-Corley dispersion relation. We use natural units in which $\hbar = k_B = c = 1$, and the metric $g_{\mu\nu}$ carries signature $(-, +, +, +)$.

II. COVARIANT LAGRANGIAN AND STRESS-ENERGY TENSOR

In this section, we first introduce a general covariant formulation of a Lagrangian describing a free scalar field with modified dispersion relation following the procedure described by Jacobson and Mattingly [16,17], and derive the corresponding energy-momentum tensor. We try to remain as general as possible with respect to the background spacetime \mathcal{M} and its metric, and defer the detailed study of a FLRW spacetime to Section II C.

A. Definitions and covariant Lagrangian

The action for a free scalar field with modified dispersion relation takes the form [16,17]

$$S_\phi = \int d^4x \sqrt{-g} (\mathcal{L}_\phi + \mathcal{L}_{\text{cor}} + \mathcal{L}_u), \quad (1)$$

where \mathcal{L}_ϕ is the standard Lagrangian of a minimally coupled free scalar field

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (2)$$

Two contributions have been added to this Lagrangian in order to introduce the modified dispersion relation. As already mentioned in the introduction, a modified dispersion relation breaks local Lorentz invariance. In such a

situation, a covariant formulation of the corresponding theory can be carried out by introducing a unit timelike vector field u^μ defining a preferred rest frame [16,17]. In Eq. (1), the first term \mathcal{L}_{cor} is responsible for the non-linear part of the dispersion relation and \mathcal{L}_u describes the dynamics of the vector field u^μ . These two corrective Lagrangians take the form

$$\mathcal{L}_{\text{cor}} = - \sum_{n,p \leq n} b_{np} (\mathcal{D}^{2n} \phi) (\mathcal{D}^{2p} \phi), \quad \mathcal{L}_u = -\lambda(g^{\mu\nu} u_\mu u_\nu + 1) - d_1 F^{\mu\nu} F_{\mu\nu}. \quad (3)$$

The tensor $F_{\mu\nu}$ is defined by $F_{\mu\nu} \equiv \nabla_\mu u_\nu - \nabla_\nu u_\mu$, where ∇_μ is the covariant derivative associated with the metric $g_{\mu\nu}$; λ is a Lagrange multiplier and the coefficients b_{np} and d_1 are arbitrary. The derivative operators \mathcal{D}^{2n} are defined further below. The overall Lagrangian maintains general covariance, which will notably ensure the conservation of the stress-energy tensor. The value of the Lagrange multiplier λ can be obtained by the extremization of the action with respect to the vector field u^μ .

Let us define more precisely the quantities appearing in the two extra Lagrangians in Eq. (3). We first assume that the spacetime \mathcal{M} is globally hyperbolic so that it can be foliated as $\mathcal{M} = \Sigma \times R$ where Σ are three dimensional spacelike hypersurfaces of constant q , where q is a scalar. It follows that the unit timelike vector field normal to these hypersurfaces is $u_\mu \equiv -(\partial_\mu q)/(-g_{\alpha\beta} \partial^\alpha q \partial^\beta q)^{1/2}$ which indeed satisfies $u_\mu u^\mu = -1$, with respect to which we will define “time” and “space” components [19,20]. Fröbenius theorem [21] guarantees that the vector field u_μ is rotation-free and thus that the field strength tensor can be written as $F_{\mu\nu} = a_\mu u_\nu - a_\nu u_\mu$, a_μ being the acceleration defined in Appendix A. As a consequence, if the vector field u_μ is geodesic then $F_{\mu\nu} = 0$. This will be for instance the case in the FLRW case but does not need to be true in general. The projector on the hypersurfaces Σ defined by

$$\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \quad (4)$$

coincides with the spatial metric as defined by an observer comoving with u_μ , since the line element can be rewritten as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(u_\mu dx^\mu)^2 + \perp_{\mu\nu} dx^\mu dx^\nu. \quad (5)$$

The covariant derivative \mathcal{D}_α associated with the induced 3-metric of any tensor field $T_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_p}$, appearing in Eq. (3) defining \mathcal{L}_{cor} , is defined as

$$\mathcal{D}_\alpha T_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_p} \equiv \perp_{\mu_1}^{\lambda_1} \dots \perp_{\mu_n}^{\lambda_n} \perp_{\sigma_1}^{\nu_1} \dots \perp_{\sigma_p}^{\nu_p} \perp_\alpha^\beta \nabla_\beta T_{\lambda_1 \dots \lambda_n}^{\sigma_1 \dots \sigma_p}, \quad (6)$$

By construction, it is the covariant derivative associated with $\perp_{\mu\nu}$ and is orthogonal to u_μ , i.e.

$$\mathcal{D}_\alpha \perp_{\mu\nu} = 0, \quad u^{\mu_i} \mathcal{D}_\alpha T_{\mu_1 \dots \mu_i \dots \mu_n}^{\nu_1 \dots \nu_p} = u_{\nu_j} \mathcal{D}_\alpha T_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_j \dots \nu_p} = u^\alpha \mathcal{D}_\alpha T_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_p} = 0. \quad (7)$$

Various identities satisfied by u_μ , $\perp_{\mu\nu}$ and \mathcal{D}_μ , which are used repeatedly in the rest of this section are given in Appendix A. We finally define the operator \mathcal{D}^{2n} appearing in the Lagrangian Eq. (3) as

$$\mathcal{D}^{2n} \equiv \mathcal{D}_{\mu_1} \mathcal{D}^{\mu_1} \dots \mathcal{D}_{\mu_n} \mathcal{D}^{\mu_n}. \quad (8)$$

For the particular case of a scalar field, this double derivative can be written as

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = \perp_\mu^\alpha \nabla_\alpha \perp_\beta^\mu \nabla^\beta \phi = \perp^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + u^\alpha \nabla_\alpha \phi \nabla_\beta u^\beta. \quad (9)$$

Throughout the paper, it will be understood that a derivative operator applies directly and only on the first term appearing on its right; derivatives of an ensemble of terms will be indicated using brackets. We will also use the shorthand notation of an overdot for the time derivative defined by an observer comoving with u^μ , i.e. $\dot{T}_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_p} \equiv u^\alpha \nabla_\alpha T_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_p}$. In particular, $\dot{\phi} \equiv u^\alpha \nabla_\alpha \phi$. The above double derivative operator can be shown to coincide with the three-dimensional Laplacian as defined by an observer comoving with u^μ [see Eq. (A5)]. The corrective Lagrangian \mathcal{L}_{cor} thus contains only “spatial” derivatives. For example, in Minkowski spacetime, this would yield plane wave solutions to the field equations with a dispersion relation for the pulsation ω^2 as a series in powers of squared momentum k^2 , where the first term in k^2 results from the free Lagrangian and the higher order terms come directly from the higher order Laplacians in \mathcal{L}_{cor} .

The stress–energy tensor is obtained by varying the action (1) with respect to the metric

$$\delta S_\phi = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}. \quad (10)$$

In the following, we derive this stress-energy tensor and extract the pressure and energy density under some hypotheses on the derivative \mathcal{D}_μ . In Appendix B, we give a detailed calculation of the stress-energy tensor in the simplest non-trivial case in which only b_{11} is non-vanishing, as considered by Jacobson and Mattingly [16,17]. The covariant derivative obviously satisfies $[\mathcal{D}^2, g_{\mu\nu}] = 0$ and we further assume that it also satisfies

$$[\mathcal{D}^2, u^\mu] = [\mathcal{D}^2, \nabla_\mu] = 0. \quad (11)$$

Although these requirements seem restrictive and may not apply in general, they are fulfilled for the relevant cosmological case of the FLRW spacetime we are interested in. We emphasize that these requirements are not necessary to the derivation of the stress-energy tensor in the case discussed in Appendix B in which only the first coefficient b_{11} is non-vanishing. The main point of the calculation is the variation of the corrective term S_{cor} to the action, which gives

$$\begin{aligned} \delta S_{\text{cor}} = & \frac{1}{2} \int \mathcal{L}_{\text{cor}} g^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d^4x \\ & - \sum_{n,p} b_{np} \int \left\{ \left[\sum_{i=0}^{n-1} \mathcal{D}^{2i} (\delta \mathcal{D}^2) \mathcal{D}^{2(n-i-1)} \phi \right] \mathcal{D}^{2p} \phi + \mathcal{D}^{2n} \phi \left[\sum_{i=0}^{p-1} \mathcal{D}^{2i} (\delta \mathcal{D}^2) \mathcal{D}^{2(p-i-1)} \phi \right] \right\} \sqrt{-g} d^4x. \end{aligned} \quad (12)$$

As a result of the commutation relations, the second integral can be rewritten through multiple integrations by parts as

$$\sum_{n,p} b_{np} (n+p) \int \left[\mathcal{D}^{2(n+p-1)} \phi \right] \delta \mathcal{D}^2 \phi \sqrt{-g} d^4x = \int E(\phi) \delta \mathcal{D}^2 \phi \sqrt{-g} d^4x, \quad (13)$$

where we used the short-hand notation $E(\phi) \equiv \sum b_{np} (n+p) \mathcal{D}^{2n+2p-2} \phi$. Using expressions (B3–B4), we finally obtain the stress-energy tensor

$$\begin{aligned} T_{\mu\nu} = & \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial_\alpha \phi \partial^\alpha \phi) g_{\mu\nu} - \sum_{n,p} b_{np} \mathcal{D}^{2n} \phi \mathcal{D}^{2p} \phi g_{\mu\nu} + 2\lambda u_\mu u_\nu \\ & + 2E \left[u_{(\mu} \nabla^\alpha u_{\nu)} \nabla_\alpha \phi - u_{(\mu} \nabla_\nu) u^\alpha \nabla_\alpha \phi - a_{(\mu} \nabla_{\nu)} \phi + \frac{1}{2} \perp_{\mu\nu} \square \phi + \frac{1}{2} g_{\mu\nu} (\ddot{\phi} + \theta \dot{\phi}) \right] \\ & - 2\nabla_{(\mu} E \nabla_{\nu)} \phi - 2\dot{E} u_{(\mu} \nabla_{\nu)} \phi - 2\dot{\phi} u_{(\mu} \nabla_{\nu)} E + \perp_{\mu\nu} \nabla_\alpha E \nabla^\alpha \phi + g_{\mu\nu} \dot{E} \dot{\phi} + T_{\mu\nu}^{(\text{F})}, \end{aligned} \quad (14)$$

where $T_{\mu\nu}^{(\text{F})} \equiv -4d_1 F_{\mu\lambda} F_\nu{}^\lambda + g_{\mu\nu} d_1 F^{\alpha\beta} F_{\alpha\beta}$. The expansion $\theta \equiv \perp^{\mu\nu} \nabla_\mu u_\nu$ is defined in Appendix A together with the acceleration a^μ . To determine the Lagrange multiplier λ , we solve the equation of motion for u_μ and obtain

$$\lambda = \frac{1}{2} \left[E (\ddot{\phi} + \theta \dot{\phi} - 2a^\alpha \nabla_\alpha \phi) - \dot{\phi} \dot{E} \right] + \lambda^{(\text{F})}, \quad (15)$$

where $\lambda^{(\text{F})} \equiv 2d_1 u^\mu \nabla^\nu F_{\nu\mu}$. Note that, as expected, $\lambda = 0$ if the dispersion relation is linear, which is obvious since in that case the action (1) does not depend on the vector field u^μ .

To conclude this part, we derive the energy density and pressure as defined by an observer comoving with u_μ ,

$$\rho \equiv u^\mu u^\nu T_{\mu\nu}, \quad p \equiv \frac{1}{3} \perp^{\mu\nu} T_{\mu\nu}, \quad (16)$$

which gives

$$\rho = \dot{\phi}^2 + \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + \sum_{n,p} b_{np} \mathcal{D}^{2n} \phi \mathcal{D}^{2p} \phi + u^\mu u^\nu T_{\mu\nu}^{(\text{F})} + 2\lambda^{(\text{F})}, \quad (17)$$

$$3p = \dot{\phi}^2 - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - 3 \sum_{n,p} b_{np} \mathcal{D}^{2n} \phi \mathcal{D}^{2p} \phi + E (3\mathcal{D}^2 \phi + a^\alpha \nabla_\alpha \phi) + \nabla_\alpha E \nabla^\alpha \phi + \dot{E} \dot{\phi} - \perp^{\mu\nu} T_{\mu\nu}^{(\text{F})}. \quad (18)$$

Although the previous expressions hold for a non-interacting scalar field, it is trivial to include a potential term in the above equations. We now turn to the particular case of the FLRW metric.

From now on, we restrict ourselves to a FLRW universe with flat spatial sections, the metric of which is given by $g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, where $a(t)$ is the scale factor and t denotes the cosmic time. The Krönecker symbol δ_{ij} represents the metric of constant time hypersurfaces in cartesian coordinates. In this particular case the expansion of the universe provides a natural definition of space and time. We thus choose the scalar function q to be the cosmic time t so that $u_\mu = -\delta_\mu^0$, $u^\mu = \delta_0^\mu$. As a consequence, we now have $F_{\mu\nu} = 0$. Also, the projector $\perp_{\mu\nu}$ takes the simple form $\perp_{\mu\nu} dx^\mu dx^\nu = a^2(t)\delta_{ij}dx^i dx^j$. Using the previous equations, it can be checked that

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = \frac{1}{a^2} \delta^{ij} \partial_i \partial_j \phi \equiv \frac{1}{a^2} \Delta \phi, \quad (19)$$

i.e. $\mathcal{D}_\mu \mathcal{D}^\mu \phi$ is the four dimensional expression of the three dimensional Laplacian, as expected from the general argument (A5). Let us note that the previous arguments can be easily extended to a FLRW universe with non-flat spatial sections. In a FLRW spacetime the commutation relations mentioned above are trivially satisfied and the energy density and pressure reduce to

$$\rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2a^2} \delta^{ij} \partial_i \phi \partial_j \phi + \sum_{n,p} b_{np} \mathcal{D}^{2n} \phi \mathcal{D}^{2p} \phi, \quad (20)$$

$$p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{6a^2} \delta^{ij} \partial_i \phi \partial_j \phi + \sum_{n,p} b_{np} \left[\frac{2}{3}(n+p) - 1 \right] \mathcal{D}^{2n} \phi \mathcal{D}^{2p} \phi, \quad (21)$$

where we have implicitly integrated by parts and discarded a total derivative term for the pressure.

We now derive the field equation for ϕ to exhibit the modified dispersion relation, and in particular to establish the link between the series contained in the Lagrangian \mathcal{L}_{cor} and the Taylor expansion that defines the dispersion relation. From now on, we use conformal time η defined by $dt \equiv a(t)d\eta$ and the reduced field $\mu \equiv a\phi$. Varying the action with respect to μ gives

$$\mu'' - \frac{a''}{a} \mu - \Delta \mu = -2a^2 \sum_{n,p} \frac{b_{np}}{a^{2(n+p)}} \Delta^{2(n+p)} \mu, \quad (22)$$

where a prime denotes a derivative with respect to η . The last term of this equation comes from the Lagrangian \mathcal{L}_{cor} which encodes all information on trans-Planckian physics. Since this term is composed only of Laplacians, it gives rise to a series in momentum, once we shift to Fourier space. For plane wave solutions of the form $\mu(\eta, \mathbf{x}) = \mu_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}$ one obtains the equation

$$\mu_k'' + \left[\omega^2(k, \eta) - \frac{a''}{a} \right] \mu_k = 0, \quad (23)$$

with the modified dispersion relation

$$\omega^2(k) = k^2 + 2a^2 \sum_{n,p} (-1)^{(n+p)} b_{np} \left[\frac{k}{a(\eta)} \right]^{2(n+p)}. \quad (24)$$

This dispersion relation can be rewritten in terms of physical pulsation $\omega_{\text{phys}} = \omega/a$ and frequency $k_{\text{phys}} = k/a$, as

$$\omega_{\text{phys}}^2(k) = k_{\text{phys}}^2 + 2 \sum_{n,p} (-1)^{(n+p)} b_{np} k_{\text{phys}}^{2(n+p)}. \quad (25)$$

Thus our procedure allows us to study the stress-energy tensor of a quantum field with dispersion relation such that ω_{phys}^2 is an analytic function of k_{phys}^2 .

We now go one step further and second-quantize the system. We indeed study the behavior of gravitational waves and density perturbations in the trans-Planckian regime, which can be reduced in the perturbative approximation to the study of a free minimally coupled quantum scalar field propagating in a FLRW classical background, see e.g. [9]. We thus write the field operator $\hat{\mu}$ (a hat denoting an operator) in the Heisenberg representation as

$$\hat{\mu}(\eta, \mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \left[\mu_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{c}_{\mathbf{k}} + \mu_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{c}_{\mathbf{k}}^\dagger \right], \quad (26)$$

where $\hat{c}_{\mathbf{k}}$ and $\hat{c}_{\mathbf{k}}^\dagger$ represent annihilation and creation operators of quanta with momentum \mathbf{k} respectively, normalized as usual through $[\hat{c}_{\mathbf{k}}, \hat{c}_{\mathbf{p}}^\dagger] = \delta(\mathbf{k} - \mathbf{p})$. The vacuum state $|0\rangle$ is defined accordingly by $\hat{c}_{\mathbf{k}}|0\rangle = 0$ and a star denotes complex conjugation. It is then easy to obtain the energy density and pressure of the scalar field ϕ in its vacuum state by inserting Eq. (26) into Eqs. (20) and (21) to get

$$\langle 0|\hat{\rho}|0\rangle = \frac{1}{4\pi^2 a^4} \int dk k^2 \left[a^2 \left| \left(\frac{\mu_k}{a} \right)' \right|^2 + \omega^2(k) |\mu_k|^2 \right], \quad (27)$$

$$\langle 0|\hat{p}|0\rangle = \frac{1}{4\pi^2 a^4} \int dk k^2 \left[a^2 \left| \left(\frac{\mu_k}{a} \right)' \right|^2 + \left(\frac{2}{3} k^2 \frac{d\omega^2}{dk^2} - \omega^2 \right) |\mu_k|^2 \right]. \quad (28)$$

The above is one of the main results of the present article, and it constitutes the basis of the discussion to follow in Section III. For the standard dispersion relation of massless quanta $\omega(k) = k$, one recovers known results, namely that in the limit $\mu_k \propto e^{-i\omega\eta}$, corresponding to subhorizon modes (see Section III), $p = \rho/3$ corresponding to a relativistic fluid, while in the limit $(\mu_k/a)' \sim 0$, corresponding to super-horizon modes (see Section III), $p = -\rho/3$. It is of interest for what follows to note that even for the standard dispersion relation, the fact that the quantity μ is frozen on superhorizon scales does not imply that the corresponding equation of state is of the cosmological constant type $p = -\rho$.

For a modified dispersion relation, as mentioned before, the energy density is the straightforward generalization of the standard expression, with $\omega^2(k)$ denoting the modified pulsation. However the pressure expression is a non-trivial generalization, and the presence of the term $d\omega^2/dk^2$ implies that various equations of state may be obtained, depending on the shape of the dispersion relation (this latter also determines the mode function μ_k).

In the following, we will also be interested in the power spectrum of the fluctuations of the field ϕ , defined by

$$\langle 0|\phi^2(\eta, \mathbf{x})|0\rangle = \int_0^{+\infty} \frac{dk}{k} k^3 P(k), \quad (29)$$

where the power spectrum per logarithmic interval $k^3 P(k)$ reads

$$k^3 P(k) = \frac{k^3}{2\pi^2} \left| \frac{\mu_k}{a} \right|^2. \quad (30)$$

In the case of power law inflation and a standard dispersion relation, the power spectrum is given by $k^3 P(k) = A_S k^{n_S-1}$. In the particular case of a de Sitter spacetime, the spectrum is scale invariant, i.e. $n_S = 1$.

III. APPLICATIONS: EQUATION OF STATE AND BACKREACTION

In this section we apply our previous calculations of the stress-energy tensor to two different situations that have arisen recently in the literature. As a first application (§ III A) we calculate the equation of state of gravitational waves of super-Planck frequencies, which have been proposed to account for the observed acceleration of the Universe [18]; our conclusions are different and do not support the claim made by these authors. In § III B and § III C we study the backreaction problem of trans-Planckian modes in inflationary cosmology [13,14]. Our calculation is indeed well suited to this case as it allows us to explicitly calculate the energy density of the fluctuations and compare it to the background energy density.

A. Trans-Planckian and dark energy

It was proposed recently that gravitational waves of super-Planck frequencies with a dispersion relation that exponentially decreases beyond the cut-off (Planck) momentum would contribute significantly to the present energy density, with an equation of state mimicking a cosmological constant [18]. To discuss this claim, let us consider the following dispersion relation

$$\omega_{\text{phys}}^2(k_{\text{phys}}) = k_{\text{phys}}^2 e^{-k_{\text{phys}}^2/k_c^2} \iff \omega^2(k) = k^2 e^{-k^2/(a^2 k_c^2)}, \quad (31)$$

where k_c is the cut-off momentum. Even though it is not exactly the dispersion relation considered in Ref. [18], it is simpler and very similar: it is linear at small momenta ($k \ll k_c$), reaches a maximum around k_c and decreases

exponentially at high momenta ($k \gg k_c$), and it will serve our purposes well enough (in addition, it was claimed that the result does not depend on the details of the dispersion relation).

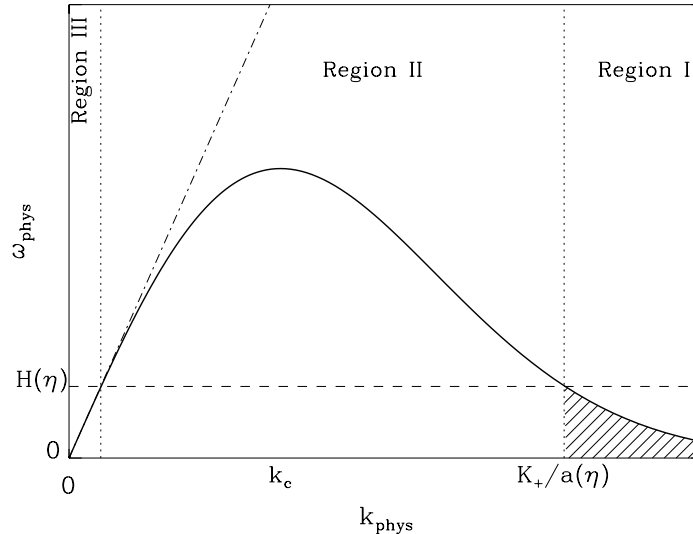


FIG. 1. Dispersion relation with an exponential decreasing tail, as envisaged in Ref. [18]. The dot-dashed line shows the linear dispersion relation, and the thick solid curve shows the modified dispersion relation. The dotted lines delimit three regions of different evolutions for the mode function: in region I, corresponding to the “tail”, a mode has trans-Planckian momentum $k > ak_c$ and sub-Hubble comoving pulsation $\omega(k) < aH$; in region II, a mode has $\omega(k) > aH$, and in region III, modes are super-horizon sized $k < aH$. The dashed area represents the “tail” modes defined by $k > K_+$ (equivalently $k_{\text{phys}} > K_+/a$ in the figure).

In Fig. 1 we show the qualitative behavior of the comoving pulsation associated to this dispersion relation as a function of comoving momentum and conformal time. There exists a region (“tail” modes) in which the modes have $k_{\text{phys}} > k_c$ and a pulsation smaller than the Hubble frequency $\omega_{\text{phys}} < H$. These modes are defined by $k > K_+ > ak_c$, where K_+ is obtained by equating ω with the comoving Hubble rate aH ,

$$K_+ \simeq ak_c \sqrt{2 \ln \frac{k_c}{H}}. \quad (32)$$

We further assume that the scale factor evolves as a power law of conformal time, *i.e.* $a(\eta) = a_0(\eta/\eta_0)^\beta$, where a_0 is the dimensionless scale factor at time η_0 , and $\beta = 1, 2, -1$ respectively for radiation domination, matter domination universe and de Sitter inflation. It is important to note that a mode that is contained in the tail a time η was already contained in the tail at any previous time; indeed it can be easily verified that the wavenumber K_+ increases with time for an expanding scale factor and constant or decreasing curvature. In the “tail” region modes have a pulsation smaller than the Hubble expansion rate, *i.e.* $\omega \ll aH = \beta/\eta$, which implies $\omega^2 \ll a''/a = \beta(\beta-1)/\eta^2$. As a consequence, the term ω^2 in the equation of motion (23) can be neglected with respect to a''/a , so that the approximate solution to Eq. (23) reads

$$\mu_k(\eta) \simeq C_+(k)a(\eta) + C_-(k)a(\eta) \int^\eta \frac{d\eta'}{a^2(\eta')}, \quad (33)$$

where C_+ and C_- are the coefficients of the growing and decaying modes respectively. These two functions are inter-related through the Wronskian normalization condition $\mu_k \mu_k^{*'} - \mu_k^* \mu_k' = i$. The solution is indeed “frozen” since for the growing mode the scalar field $\phi = \mu_k/a$ is constant in time. The claim of Ref. [18] is based on the assumption that frozen modes have an equation of state of a cosmological constant type $p = -\rho$. As already mentioned above, this is not true even if the dispersion relation is standard and the mode is frozen (super-horizon sized), since the equation of state then reads $p = -\rho/3$. In addition, as we argue below, this claim does not hold either when the dispersion relation is modified.

In order to calculate the energy density and pressure, and the relation between these two quantities, it is necessary to specify the vacuum state. Unfortunately, for the “tail” modes, it cannot be chosen unambiguously since the WKB approximation breaks down in the limit $\omega \ll aH$ and the concept of adiabatic vacuum cannot be used. This ambiguity also implies that the initial value of μ , i.e. the coefficients $C_-(k)$ and $C_+(k)$ cannot be determined unambiguously. This problem has not been addressed in Ref. [18] as only one branch of the solution to the equation of motion is considered and the normalization coefficient (similar to the above C_\pm) is taken to be independent of k . In any case the solution given in Ref. [18] is not a correct solution to the field equation, since these authors neglect the contribution of the term a''/a in the limit $\eta \rightarrow -\infty$, but a direct comparison between the various terms in their Eq. (22) shows that instead the term a''/a dominates in this limit, in agreement with the above discussion.

In the following, we propose to circumvent the ambiguity related to the definition of a correct vacuum state by discussing two possible generic sets of initial conditions.

1. Power-law initial conditions

The above problem is in a certain sense similar to the situation encountered in cosmology before the advent of inflation. Indeed in the absence of an inflationary epoch, quantum modes can only enter the horizon as time goes, and the initial data had to be specified on scales larger than the horizon in a regime where the field is frozen. In contrast inflation has the virtue of stretching modes beyond the horizon, so that the initial data can be specified without ambiguity while the mode lies well inside the horizon. In the absence of such a mechanism, it had been proposed to use a universal power law to describe the power spectrum of the initial fluctuations, an approach that relied on astrophysical arguments. We thus propose to adopt a similar choice and take the growing mode coefficient to be a power law in momentum

$$C_+(k) = k_I^{-1/2} \left(\frac{k}{k_I} \right)^\alpha, \quad (34)$$

where k_I is a constant related to the amplitude of the fluctuations and α a constant. It follows from Eqs. (27–28) that the energy density and pressure of the modes in the tail at a time η are

$$\langle 0|\hat{\rho}|0\rangle = \frac{k_I^4}{4\pi^2 a^2} \int_{K_+/k_I}^\infty u^{4+2\alpha} e^{-u^2 \lambda^2} du, \quad (35)$$

$$\langle 0|\hat{p}|0\rangle = -\frac{k_I^4}{12\pi^2 a^2} \int_{K_+/k_I}^\infty u^{4+2\alpha} [1 + 2\lambda^2 u^2] e^{-u^2 \lambda^2} du, \quad (36)$$

where we have introduced the dimensionless parameter λ which depends on the time at which the density and pressure are evaluated $\lambda \equiv k_I/(ak_c)$. This can be integrated to give

$$\langle 0|\hat{\rho}|0\rangle = \frac{(ak_c)^4}{8\pi^2 a^2} \left(\frac{ak_c}{k_I} \right)^{2\alpha+1} \Gamma\left(\alpha + \frac{5}{2}, \frac{K_+^2}{a^2 k_c^2}\right), \quad (37)$$

$$\langle 0|\hat{p}|0\rangle = -\frac{(ak_c)^4}{8\pi^2 a^2} \left(\frac{ak_c}{k_I} \right)^{2\alpha+1} \left[\frac{1}{3} \Gamma\left(\alpha + \frac{5}{2}, \frac{K_+^2}{a^2 k_c^2}\right) + \frac{2}{3} \Gamma\left(\alpha + \frac{7}{2}, \frac{K_+^2}{a^2 k_c^2}\right) \right], \quad (38)$$

where $\Gamma(\beta, x) \equiv \int_x^\infty dt e^{-t} t^{\beta-1}$ is the incomplete gamma function [22]. The equation of state parameter is thus obtained as (note that it does not depend on the arbitrary wave number k_I)

$$\omega \equiv \frac{p}{\rho} = -\frac{1}{3} - \frac{2}{3} \frac{\Gamma\left[\alpha + 7/2, K_+^2/(a^2 k_c^2)\right]}{\Gamma\left[\alpha + 5/2, K_+^2/(a^2 k_c^2)\right]}, \quad (39)$$

and since $K_+ \gg ak_c$, one can use the large argument expansion of the incomplete gamma function $\Gamma(\beta, x) \simeq x^{\beta-1} \exp(-x)$ at large x to derive the following expression

$$\omega \simeq -\frac{1}{3} - \frac{2}{3} \alpha - \frac{4}{3} \ln\left(\frac{k_c}{H}\right), \quad (40)$$

where we have used Eq. (32) to express $K_+/(ak_c)$. If k_c is of order of the Planck scale, with $H_0/M_{\text{Pl}} \simeq 10^{-61}$, then one obtains today

$$\omega \simeq -186. \quad (41)$$

This value does not depend on the normalization scale k_I and is in clear contradiction with a cosmological constant type equation of state, i.e. $p \neq -\rho$. It is rather a “phantom energy component” according to the terminology introduced in Ref. [23], i.e. a component for which $\omega < -1$. In Ref. [23], observational constraints on the equation of state have been studied, and $\omega \simeq -186$ already appears in contradiction with the SNIa data, see Fig. 6 of Ref. [23]. Furthermore, if one evaluates the energy density contained in the “tail” modes today (using again the asymptotic value of the gamma function), one finds

$$\rho_{\text{tail}} \sim H^2 k_c^2 \left(\frac{K_+}{k_I} \right)^{2\alpha+1}, \quad (42)$$

or if we rather calculate $\Omega_{\text{tail}} \equiv \rho_{\text{tail}}/\rho_{\text{crit}}$, where $\rho_{\text{crit}} \equiv 3H^2 M_{\text{Pl}}^2/8\pi$ is the critical energy density, one obtains

$$\Omega_{\text{tail}} \simeq 0.1 \left(\frac{k_c}{M_{\text{Pl}}} \right)^2 \left(\frac{K_+}{k_I} \right)^{2\alpha+1}. \quad (43)$$

Therefore, if $k_c \sim M_{\text{Pl}}$, $\Omega_{\text{tail}} \sim 1$ if $\alpha = -1/2$ and/or $k_I = K_+ \simeq 16.76 M_{\text{Pl}}$. But the important point is that this can be obtained only at the expense of fine-tuning of the parameters, and the above choice does not seem natural; actually, the parameter k_I is arbitrary and *a priori* there exists no natural way to fix it.

2. Minimising energy

A second possibility is to fix the initial conditions by requiring that the initial state configuration minimizes the energy, as advocated in Ref. [24] and used in Refs. [2,3]. To this end we parametrize μ'_k as $\mu'_k/\mu_k \equiv x + iy$, where x and y are two real functions, and we rewrite the energy density as

$$\rho = \frac{1}{8\pi a^4} \int dk k^2 \frac{1}{y} [\mathcal{H}^2 + x^2 + y^2 - 2\mathcal{H}x + \omega^2] \quad (44)$$

with $\mathcal{H} \equiv aH$. One can now find the extrema of this expression under the Wronskian normalization constraint to obtain the mode function and its derivative at the initial time η_i

$$|\mu_k(\eta_i)| = \frac{1}{\sqrt{2\omega(k, \eta_i)}}, \quad \mu'_k(\eta_i) = [\mathcal{H} + i\omega(k, \eta_i)]\mu_k(\eta_i). \quad (45)$$

It follows that in the “tail” region, the field μ evolves in time as

$$\mu_k^{[\text{I}]}(\eta) = \frac{1}{\sqrt{2\omega(k, \eta_i)}} \frac{a(\eta)}{a(\eta_i)} \left\{ 1 + i\omega(k, \eta_i) \int_{\eta_i}^{\eta} \left[\frac{a(\eta)}{a(\eta_i)} \right]^2 d\eta \right\}. \quad (46)$$

The energy of the “tail” modes at a later time η can then be estimated as

$$\rho^{[\text{I}]}(\eta) \simeq \frac{1}{8\pi^2 a^2(\eta_i) a^2(\eta)} \int_{K_+}^{\infty} dk k^2 \frac{\omega^2(k, \eta)}{\omega(k, \eta_i)}. \quad (47)$$

It can then be checked that this quantity depends strongly on the choice of the initial time. Moreover, since the dispersion relation decreases exponentially fast as $k \rightarrow +\infty$ and $\eta_i < \eta$, the integral diverges exponentially when $k \rightarrow \infty$.

In spite of the ambiguity related to the choice of the initial state, the above two solutions have the merit to show that there is no rigorous argument in favor of the claim made by Mersini *et al.* [18]. In particular, one does not generically find (or expect) the “frozen” modes of the “tail” neither to have a cosmological constant type equation of state nor to have an energy density today coinciding naturally with the critical energy density. As we have mentioned above, the difference with the results obtained in Ref. [18] presumably lies in an incorrect assumption made by these authors to derive the time evolution of the mode functions. One should also add that these authors [18] have not addressed the problem of the energy contained in modes with momentum $k < K_+$ (in particular region II in Fig. 1). Indeed if one considers the “tail” modes as a source of gravitational energy today, there is *a priori* no reason to discard the contribution of other modes. However the subhorizon non-frozen modes oscillate, and it is easy to see that their energy density will be of order M_{Pl}^4 , which is nothing less than the celebrated long-standing problem of the cosmological constant.

The previous section has briefly touched upon the issue of backreaction in trans-Planckian cosmology, which arises whenever the energy density of the fluctuations (scalar or tensor) becomes comparable to the background energy density. In this case, the perturbative approximation used to describe the evolution of the mode functions breaks down, and one has to resort to a second order (complex) treatment of the Einstein equations [25].

Backreaction effects may arise in different situations in inflationary cosmology. Liddle and Lyth [26] (in collaboration with E. Stewart) pointed out that if the modes of comoving wavelength of order of the horizon size, i.e. modes of cosmological interest for structure formation, start in a non-vacuum initial state, then inflation suffers from a very important backreaction problem. It is interesting for the following discussion to summarize briefly their argument. A mode with momentum k exits the horizon $N_e(k) \sim 56 - \ln(k/H_0)$ e -folds before the end of inflation [27]. At horizon crossing, this mode carries a physical momentum $k_{\text{phys}} = H_{\text{inf}}$, $H_{\text{inf}} \sim 10^{13} \text{ GeV}$ being the Hubble scale during inflation. Therefore, this mode had a physical momentum $k_{\text{phys}} = \exp(N_i) H_{\text{inf}}$ at the onset of inflation, $N_i = N_{\text{tot}} - N_e$ being the number of e -folds of inflation before horizon crossing, and N_{tot} the total number of e -folds of inflation. There is no reason to expect that inflation lasted just sufficiently long so as to stretch beyond the horizon only those modes with wavelength smaller than the horizon size today, and in general $N_{\text{tot}} \gg N_e$ hence $N_i \gg 1$. The energy density contained in these modes at the beginning of inflation is thus $\delta\rho_k \sim (2\pi^2)^{-1} n_k \exp(4N_i) H_{\text{inf}}^4$ up to a numerical factor of order unity, with n_k the occupation number (we discard the zero-point energy). The ratio of this energy density to the critical energy density thus reads $\delta\rho/\rho_{\text{crit}} \sim n_k \exp(4N_i) (H_{\text{inf}}/M_{\text{Pl}})^2$; finally $H_{\text{inf}} \sim 10^{-6} M_{\text{Pl}}$ implies the very stringent constraint $n_k \lesssim \exp(-4N_i + 28)$ to avoid backreaction problems. In other words, either $N_i \lesssim 7$, which may seem unnatural, or $n_k \ll 1$ which implies that these modes are in their vacuum state or very close to it. Even though this argument was developed in the context of standard inflationary cosmology without modified dispersion relation and trans-Planckian physics, it can be carried over directly to our present problem.

In fact, this has been done recently by Tanaka [13] who assumed that the modification of the dispersion relation could be represented by a non-vacuum initial state in a theory with unmodified dispersion relation. This implied, in much the same way as above, that backreaction effects should be very important. Starobinsky [14] has further argued that if adiabaticity is broken at some point in the evolution of a given mode, then at late times, when adiabaticity is restored, this mode behaves as the sum of positive and negative energy plane waves, meaning that a finite amount of quanta has been created, and the energy density they carry is large compared to the critical energy density today. However neither of these authors had at their disposal a rigorous expression for the energy density and they could not provide a quantitative estimate of this backreaction energy density. Nevertheless as we show in the following their conclusions remain correct, notably because the correct expression for the energy density Eq. (27) is the generalization of the standard expression. The overall picture for the origin and magnitude of backreaction can be described as follows.

Consider a general dispersion relation such that for a given comoving wavenumber k , for time $\eta_2(k) < \eta < \eta_3(k)$, where $\eta_3(k) \simeq -1/k$ denotes the conformal time of horizon crossing, the WKB approximation is valid, i.e. $\omega'/\omega^2 \ll 1$. This notably implies $\omega(k) \gg \mathcal{H}$ for $\eta_2(k) < \eta < \eta_3(k)$, as the WKB condition would not be satisfied otherwise. In this case, the mode function evolves according to

$$\mu_k(\eta) \simeq \frac{\alpha(k)}{\sqrt{2\omega(k, \eta)}} e^{-i \int^\eta \omega(k, \eta') d\eta'} + \frac{\beta(k)}{\sqrt{2\omega(k, \eta)}} e^{i \int^\eta \omega(k, \eta') d\eta'}, \quad (48)$$

with $|\alpha(k)|^2 - |\beta(k)|^2 = 1$ from the Wronskian normalization condition. In principle the Bogoliubov coefficients $\alpha(k)$ and $\beta(k)$ depend on time, but their time dependence can be neglected to first order in the WKB expansion. These coefficients can be obtained by matching the mode function μ_k and its first derivative at time $\eta_2(k)$ with the solution of the field equation in the region $\eta < \eta_2(k)$. If the WKB approximation was also valid at all times $\eta < \eta_2(k)$, and the Bunch-Davies adiabatic vacuum is chosen as the initial state of the field (in mode k at least), then $\beta_k \approx 0$ and $\alpha_k \approx 1$ at time η . Furthermore, the modification to the power spectrum in this case is of first order in β_k , and thus remains small [12]. However, if at some time prior to $\eta_2(k)$, the WKB condition was violated, then $|\beta_k|$ can be *a priori* large; this had been remarked by Starobinsky [14]. This may induce large modifications in the power spectrum, but it also represents the creation of a large amount of energy density due to the breaking of adiabaticity. In effect, the energy density contained at time η in all modes such that the mode function $\mu_k(\eta)$ is given by Eq. (48) can be written as

$$\langle 0|\hat{\rho}|0\rangle(\eta) = \frac{1}{4\pi^2 a^4} \int dk k^2 \left\{ \frac{1}{2\omega} \left[\omega^2 + |\gamma|^2 \right] + \frac{|\beta_k|^2}{\omega} \left[\omega^2 + |\gamma|^2 \right] + \frac{\alpha_k \beta_k^*}{2\omega} \left[\omega^2 + \gamma^2 \right] e^{-2i \int^\eta \omega(k, \eta') d\eta'} \right. \\ \left. + \frac{\alpha_k^* \beta_k}{2\omega} \left[\omega^2 + (\gamma^*)^2 \right] e^{2i \int^\eta \omega(k, \eta') d\eta'} \right\}, \quad (49)$$

where we have used $|\alpha_k|^2 = 1 + |\beta_k|^2$. In the above expression, the quantity γ is defined as follows

$$\gamma(k, \eta) \equiv \left[\frac{\omega'(k, \eta)}{2\omega(k, \eta)} + i\omega(k, \eta) + \frac{a'}{a} \right]. \quad (50)$$

In a situation where WKB is a good approximation, we have $\gamma/\omega \simeq i$ and the previous expression reduces to

$$\langle 0|\hat{\rho}|0\rangle = \frac{1}{4\pi^2 a^4} \int dk k^2 \left(\frac{1}{2} + |\beta_k|^2 \right) \omega(k). \quad (51)$$

Note that in order to remove the two oscillatory terms, no procedure of time averaging is needed in contrast with what has been done in Ref. [13]. If $\omega(k)$ does not decrease faster than k^{-3} as $k \rightarrow +\infty$, this integral diverges as in flat space due to the first term in the integrand. We interpret this infinite quantity as the zero-point energy which we subtract (adiabatic regularization), and interpret the remainder as the presence of finite energy density in modes with occupation number $|\beta_k|^2$, as usual. It is interesting to note that if $\omega(k)$ decreases faster than k^{-3} as $k \rightarrow +\infty$, the integral is no longer divergent, and delicate questions on the necessity of renormalization arise. We will not touch upon these subtle issues in the present article, and will adopt the simplistic point of view in which the zero-point energy is subtracted in all cases. In Ref. [13], an expression similar to Eq. (51) above had been used to discuss the magnitude of backreaction, but the origin of β_k had been left unspecified. The above discussion establishes the link that was missing and it also justifies more rigorously the approach of Refs. [13,14], i.e. it shows explicitly that having a non standard dispersion relation is equivalent to considering non-vacuum quantum states for the perturbations as guessed in these studies. Furthermore, it also shows how to calculate β_k , i.e. through the matching with the solution to the mode equation at time $\eta < \eta_2(k)$.

In the following, we provide a concrete example of such a calculation and discuss the magnitude of β_k for a general class of dispersion relation. The WKB approximation is valid whenever $|\omega'/\omega^2| \ll 1$, or

$$\left| \frac{H}{\omega_{\text{phys}}} \left(1 - \frac{d \ln \omega_{\text{phys}}}{d \ln k_{\text{phys}}} \right) \right| \ll 1. \quad (52)$$

The WKB approximation can thus be violated in two general ways: by space-time curvature effects, i.e. when $\omega_{\text{phys}} \ll H$, and/or by singularities in the dispersion relation or its first derivative. In the following we will be interested only in the former class. An illustrative example of the latter is given by the Jacobson-Corley dispersion relation that becomes complex, which has already been discussed extensively [2,3]. We note that the dispersion relation considered in Section III A violates the WKB approximation at large physical momenta in both ways, namely $d \ln \omega_{\text{phys}}/d \ln k_{\text{phys}}$ diverges as $k_{\text{phys}} \rightarrow +\infty$ (or $\eta \rightarrow -\infty$), and ω_{phys} decreases much faster than H as $\eta \rightarrow -\infty$. In order to compute the Bogoliubov coefficient β_k one needs to solve the mode equation in the region where the WKB approximation is violated. However, as discussed in Section III A, in this region one cannot determine unambiguously a vacuum state, and in particular one cannot fix unambiguously initial data. We thus make a further assumption, and assume that the dispersion relation is such that the WKB approximation is restored in the far past $\eta \rightarrow -\infty$. More precisely we assume that for a given comoving wavenumber k , there exists a time $\eta_1(k)$ such that for $\eta < \eta_1(k)$, the WKB condition is satisfied; this allows us to define proper initial data for the mode evolution.

C. Backreaction: examples

A dispersion relation with the above property can be constructed easily. For instance, consider the lowest order generalization of the Corley-Jacobson dispersion relation, i.e. $\omega_{\text{phy}}^2 = k_{\text{phys}}^2 + 2b_{11}k_{\text{phys}}^4 - 2b_{12}k_{\text{phys}}^6$. Depending on the sign of the coefficients b_{11} and b_{12} , this dispersion relation can present both a maximum around k_c and a minimum at physical momentum larger than k_c . For a specific choice of these coefficients, this minimum can be chosen to be smaller than the Hubble scale at some time η , as depicted in Fig. 2.

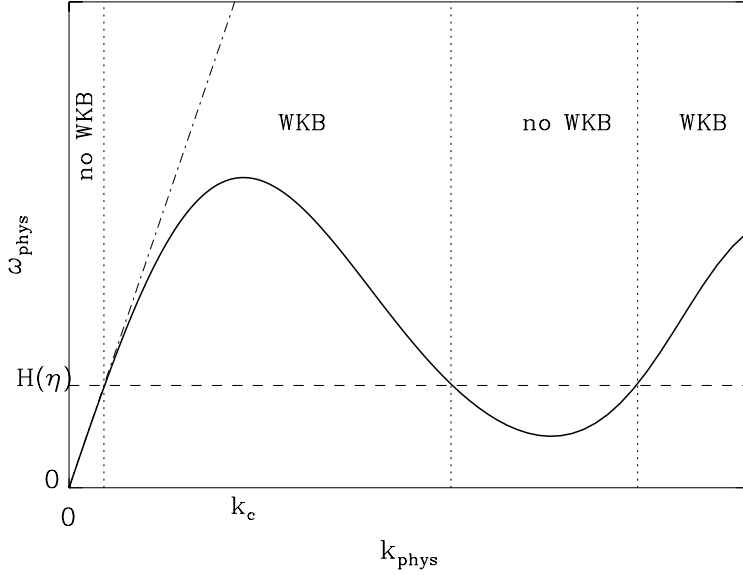


FIG. 2. Example of a modified dispersion relation which breaks the WKB approximation in the trans-Planckian regime when $\omega_{\text{phys}} < H$, as indicated.

This is only an example of the desired class of dispersion relations, and the following discussion remains general with respect to the form of $\omega_{\text{phys}}(k_{\text{phys}})$. We fix the initial conditions in the region $\eta < \eta_1(k)$ where the WKB approximation holds by selecting positive frequency modes, which corresponds to the choice of the Bunch-Davies adiabatic vacuum state for the field,

$$\mu_k^{[0]}(\eta) = \frac{1}{\sqrt{2\omega(k, \eta)}} e^{-i \int_{\eta_i}^{\eta} \omega(k, \eta') d\eta'}. \quad (53)$$

At time $\eta_1(k)$, this mode enters region I where $\omega(k) \ll \mathcal{H}$, and μ_k and its first derivative must be matched to the solution Eq. (33), which gives

$$\mu_k^{[\text{I}]}(\eta) = \frac{1}{\sqrt{2\omega[k, \eta_1(k)]}} e^{-i \int_{\eta_i}^{\eta_1(k)} \omega(k, \eta') d\eta'} \frac{a(\eta)}{a(\eta_1)} \left\{ 1 - \gamma(k, \eta_1) \int_{\eta_1(k)}^{\eta} \left[\frac{a(\eta_1)}{a(\eta')} \right]^2 d\eta' \right\}, \quad (54)$$

where the quantity γ has been defined previously, see Eq. (50). Note that γ_1 carries dimensions of inverse time and in particular, for general \mathcal{C}^1 dispersion relations, $\gamma_1 \eta_1$ is a number of order unity, which depends weakly on k [see Eq. (60) below]. At time $\eta_2(k)$, the mode enters region II in which the WKB approximation holds, i.e. $\omega(k) \gg \mathcal{H}$, and μ_k and its first derivative must be matched to the WKB solution, giving

$$\mu_k^{[\text{II}]}(\eta) = \frac{1}{\sqrt{2\omega(k, \eta)}} \left[\alpha_k e^{-i \int_{\eta_2(k)}^{\eta} \omega(k, \eta') d\eta'} + \beta_k e^{+i \int_{\eta_2(k)}^{\eta} \omega(k, \eta') d\eta'} \right], \quad (55)$$

and the two coefficients α_k and β_k are explicitly given by

$$\alpha_k = \frac{i}{2} \sqrt{\frac{\omega[k, \eta_1(k)]}{\omega[k, \eta_2(k)]}} \frac{e^{-i \int_{\eta_i}^{\eta_1(k)} \omega(k, \eta') d\eta'}}{\omega[k, \eta_1(k)]} \left[\frac{a_2}{a_1} \gamma_2^* - \frac{a_1}{a_2} \gamma_1 - a_1 a_2 \gamma_1 \gamma_2^* \int_{\eta_1(k)}^{\eta_2(k)} \frac{d\eta'}{a^2} \right], \quad (56)$$

$$\beta_k = -\frac{i}{2} \sqrt{\frac{\omega[k, \eta_1(k)]}{\omega[k, \eta_2(k)]}} \frac{e^{-i \int_{\eta_i}^{\eta_1(k)} \omega(k, \eta') d\eta'}}{\omega[k, \eta_1(k)]} \left[\frac{a_2}{a_1} \gamma_2 - \frac{a_1}{a_2} \gamma_1 - a_1 a_2 \gamma_1 \gamma_2 \int_{\eta_1(k)}^{\eta_2(k)} \frac{d\eta'}{a^2} \right], \quad (57)$$

and where γ_2 is defined as in Eq. (50) albeit it is evaluated at time $\eta_2(k)$. The previous equations give the general expressions of the coefficients α_k and β_k in region II. These expressions can be further worked out in the following two situations: (i) when $|\eta_1| \gg |\eta_2|$ and the scale factor is written as $a(\eta) \equiv a_{\text{inf}}(\eta/\eta_{\text{inf}})^\beta$ or (ii) when $\eta_2 \sim \eta_1$.

Let us start with the first situation. Assuming $|\eta_1| \gg |\eta_2|$ and using $\omega_2\eta_2 = \omega_1\eta_1 = \sqrt{\beta(\beta-1)}$, the leading contribution to the two coefficients α_k and β_k can be written as

$$\alpha_k \simeq -\frac{i}{2} \frac{e^{-i \int_{\eta_1}^{\eta_1(k)} \omega(k, \eta') d\eta'}}{\sqrt{\beta(\beta-1)}} \gamma_2^* |\eta_2| \left(1 + \frac{\gamma_1 |\eta_1|}{1-2\beta}\right) \left(\frac{\eta_2}{\eta_1}\right)^{\beta-1/2}, \quad (58)$$

$$\beta_k \simeq \frac{i}{2} \frac{e^{-i \int_{\eta_1}^{\eta_1(k)} \omega(k, \eta') d\eta'}}{\sqrt{\beta(\beta-1)}} \gamma_2 |\eta_2| \left(1 + \frac{\gamma_1 |\eta_1|}{1-2\beta}\right) \left(\frac{\eta_2}{\eta_1}\right)^{\beta-1/2}, \quad (59)$$

The values of $\gamma_1\eta_1$ and $\gamma_2\eta_2$ can be cast in the form

$$(\gamma\eta)_{1,2} = \frac{3}{2}\beta + i\beta(\beta-1) - \frac{1}{2}\beta \frac{d \ln \omega_{\text{phys}}}{d \ln k_{\text{phys}}} \Big|_{1,2}, \quad (60)$$

which shows that these quantities are of order unity and do not depend strongly on k unless the dispersion relation contains exponential factors important at the time of matching. It follows that in region II, one has

$$|\mu_k^{[\text{II}]}(\eta)|^2 \simeq \frac{1}{4\omega(k, \eta)\beta(\beta-1)} \left(\frac{\eta_2}{\eta_1}\right)^{2\beta-1} |\gamma_2\eta_2|^2 \left|1 + \frac{\gamma_1 |\eta_1|}{1-2\beta}\right|^2 \left\{1 - \cos\left[2\varphi_2 + 2 \int_{\eta_2(k)}^{\eta} \omega(k, \eta') d\eta'\right]\right\}, \quad (61)$$

where $\varphi_2 \equiv \arg(\gamma_2)$. In region III, in which the mode exits the horizon on the linear part of the dispersion relation, the solution reads $\mu_k(\eta) \simeq C_{\text{III}}(k)a(\eta)$. The constant $C_{\text{III}}(k)$ is obtained by matching the super-horizon solution with the solution in region II given above and it essentially determines the power spectrum [see Eq. (30)]. When the mode k exits the zone II and enters the zone III, the dispersion relation is linear $\omega \simeq k$. We deduce that $\eta_3(k) = \sqrt{\beta(\beta-1)}/k$ and thus that

$$\begin{aligned} k^3 P(k) &= \frac{\eta_{\text{inf}}^{2\beta}}{8\pi^2 a_{\text{inf}}^2} [\beta(\beta-1)]^{-\beta-1} |\gamma_2\eta_2|^2 \left|1 + \frac{\gamma_1 |\eta_1|}{1-2\beta}\right|^2 k^{2\beta+2} \left(\frac{\eta_2}{\eta_1}\right)^{2\beta-1} \\ &\times \left\{1 - \cos\left[2\varphi_2 + 2 \int_{\eta_2(k)}^{\sqrt{\beta(\beta-1)}/k} \omega(k, \eta') d\eta'\right]\right\}. \end{aligned} \quad (62)$$

Two interesting features should be noted on this form of the power spectrum. First, the standard spectral index of the overall amplitude $n_S - 1 = 2\beta + 2$ is modified due to the factor $(\eta_2/\eta_1)^{2\beta-1}$ which *a priori* depends on k . Second, the power spectrum exhibits superimposed oscillations and therefore it can *a priori* vanish for some values of k . The above formula can be used to study the features of these oscillations once the dispersion relation has been specified but we will pursue this goal here. However, in the particular case of a de Sitter inflationary period ($\beta = -1$), the previous features are no longer present since all quantities are functions of $k\eta$ only. Notably, since H is a constant in de Sitter space, the two solutions η_1 and η_2 to the equation $\omega_{\text{phys}} = H$ read $\eta_1 = C_1/k$ and $\eta_2 = C_2/k$, where C_1 and C_2 are constants, and therefore η_2/η_1 is independent of k . For the same reason, since $\omega(k, \eta) = \omega(k\eta)$, a change of variables $\eta \rightarrow u \equiv k\eta$ in the integral appearing in Eq. (62) shows that this quantity does not depend on k . The spectral index is thus unchanged, i.e. $n_S = 1$, and the superimposed oscillations reduce to a constant numerical factor. The only dependence on k comes through the factors $d \ln \omega_{\text{phys}}/d \ln k_{\text{phys}}$ in $\gamma_1\eta_1$ and $\gamma_2\eta_2$, which as mentioned previously, is weak.

Finally, the energy density contained in the modes in region II at time η can be written as:

$$\rho(\eta) \simeq \frac{1}{16\pi^2 \beta(\beta-1)a^4} \int_{\mathcal{K}} dk k^2 \left(\frac{\eta_2}{\eta_1}\right)^{2\beta-1} |\gamma_2\eta_2|^2 \left|1 + \frac{\gamma_1 |\eta_1|}{1-2\beta}\right|^2 \omega(k, \eta), \quad (63)$$

and the domain of integration \mathcal{K} is such that $aH < k < ak_2$ with k_2 being the smallest wavenumber such that $\omega_{\text{phys}}(k_2) = H$ and $k_2 > H$. It is artificial to consider different domains of wavenumbers and to treat the corresponding contributions to the energy density separately, but those modes with wavenumbers $k > ak_1$ [where k_1 is the other wavenumber $> H$ such that $\omega_{\text{phys}}(k_1) = H$] are in their vacuum state and WKB holds so the contribution to the integral has been removed, while those with $ak_1 < k < ak_2$ are in the non-WKB zone, where it is ambiguous to define a vacuum state and to calculate the contribution to the energy density. On the other hand, it is sufficient to show that one of these contributions is of the order of the background energy density to demonstrate that there is a backreaction problem. This is the spirit of the following calculation and the interval chosen is particularly well-suited since in the standard case the corresponding contribution vanishes. In the present case, one obtains for the case $n_S = 1$

$$\rho(\eta) \sim \left(\frac{\eta_1}{\eta_2}\right)^3 \int_{\mathcal{K}} dk_{\text{phys}} k_{\text{phys}}^2 \omega_{\text{phys}}(k) = \left(\frac{\eta_1}{\eta_2}\right)^3 \mathcal{O}(k_c^4), \quad (64)$$

where we have approximated the integral to the peak value of ω_{phys} at k_c . This value should be compared with the background energy density during inflation $\rho_{\text{inf}} = M_{\text{Pl}}^2 H_{\text{inf}}^2$. Therefore, if we take $k_c \simeq M_{\text{Pl}}$, we see that $\rho \gg \rho_{\text{inf}}$ due to $H_{\text{inf}} < M_{\text{Pl}}$ and $\eta_1/\eta_2 \gg 1$, the latter equation expressing the fact that the coefficient β_k is large in region II. This result coincides with the analysis of Refs. [13,14].

The other situation where the spectrum and the energy density can be evaluated explicitly if the time η_2 is close to η_1 . If we write $\eta_2 = \eta_1(1 + \epsilon)$, where ϵ is a small parameter, then one finds for the two coefficients α_k and β_k [see Eqs. (56, 57)]

$$\alpha_k \simeq \left\{ 1 - \epsilon \frac{i\omega_1\eta_1}{2} \left[1 + \frac{Q_1}{\omega_1^2} - \frac{\kappa a_1^2}{2\omega_1^2} \left(\frac{\rho_0}{3} - p_0 \right) \right] \right\} e^{-i \int_{\eta_1}^{\eta_1(k)} \omega(k, \eta') d\eta'} + \mathcal{O}(\epsilon^2), \quad (65)$$

$$\beta_k \simeq -\epsilon \frac{i\omega_1\eta_1}{2} \left[1 - \frac{Q_1}{\omega_1^2} + \frac{\kappa a_1^2}{2\omega_1^2} \left(\frac{\rho_0}{3} - p_0 \right) \right] e^{-i \int_{\eta_1}^{\eta_1(k)} \omega(k, \eta') d\eta'} + \mathcal{O}(\epsilon^2), \quad (66)$$

where ρ_0 and p_0 are the background energy density and pressure respectively. The coefficient Q is defined by $Q \equiv 3\omega'^2/(4\omega^2) - \omega''/(2\omega)$. The interpretation of these formulae is as follows. To leading order in ϵ , the coefficient α_k is just a pure phase and the coefficient β_k vanishes as expected. Two terms appear to the next order in ϵ where the corrections show up. The first is Q/ω^2 . The condition for WKB to be valid is $|Q/\omega^2| \ll 1$ and therefore this term indicates the magnitude of violation of the WKB approximation. The second term is proportional to $\rho_0/3 - p_0$. The presence of this term is also natural because it vanishes for radiation ($a'' = 0$), in which case the exact solution for the mode function $\mu_k(\eta)$ is a complex exponential and no correction is expected for the overall amplitude. Finally, repeating the same calculations as above for the power spectrum, one finds

$$k^3 P(k) = \frac{\eta_{\text{inf}}^{2\beta}}{2\pi^2 a_{\text{inf}}^2} [\beta(\beta - 1)]^{-\beta} k^{2\beta+2} \times \left\{ 1 - \epsilon \frac{\omega_1\eta_1}{2} \left[1 + \frac{Q_1}{\omega_1^2} + \frac{\kappa a_1^2}{2\omega_1^2} \left(\frac{\rho_0}{3} - p_0 \right) \right] \sin \left[2 \int_{\eta_2(k)}^{\sqrt{\beta(\beta-1)/k}} \omega(k, \eta') d\eta' \right] \right\} + \mathcal{O}(\epsilon^2). \quad (67)$$

The correction to the power spectrum is of order ϵ as expected. To this order, as before, we have a modified overall amplitude and superimposed oscillations appear. We can also estimate the energy density. Inserting the expression of the coefficient β_k into Eq. (51), one obtains

$$\rho \sim \mathcal{O}(\epsilon^2 k_c^4). \quad (68)$$

Therefore, there is no backreaction problem if $\epsilon^2 k_c^4 \lesssim M_{\text{Pl}}^2 H_{\text{inf}}^2$. Let us write k_c as $k_c \equiv 10^{-s} M_{\text{Pl}}$, where the coefficient s fixes the scale of the characteristic wavenumber with respect to the Planck mass. One can also write ϵ as $\epsilon \equiv 10^{-p}$, where p roughly gives the order of magnitude of the modification to the power spectrum. If we take $H_{\text{inf}} = 10^{-6} M_{\text{Pl}}$, then there is no backreaction problem if $p + 2s \gtrsim 6$ that is to say if $s \gtrsim 3 - p/2$. Therefore a modification of the spectrum of order 10% in principle already detectable now with COBE or in the near future with MAP can exist without a significant backreaction problem provided the characteristic scale $k_c \lesssim 10^{-2.5} M_{\text{Pl}}$ (but larger than the Hubble scale of inflation). Similarly, a modification of 1% in principle observable by the Planck satellite mission can be obtained if $s \gtrsim 2$. Interestingly, these domains encompass the Grand Unification scale $\sim 10^{16}$ GeV and possibly the string scale.

At this stage one should note that the issue of backreaction in the present Universe [14], which can be interpreted as the production of gravitons of super-Planck momentum, applies to those dispersion relations which violate WKB today, *i.e.* those for which $\omega < a_0 H_0$ today (notwithstanding singularities in the dispersion relation). Since the comoving Hubble scale today is orders of magnitude below the comoving Hubble scale of inflation, a dispersion relation which broke the WKB approximation during inflation does not necessarily imply production of gravitons today. This holds in particular for those dispersion relations above which entail a modification of the power spectrum without a strong backreaction problem at the time of inflation.

Finally one should note that the above situation may be encountered for a wider class of dispersion relations, if the early time evolution of the scale factor is different from inflationary expansion. Notably consider a spacetime which is asymptotically Minkowski as $\eta \rightarrow -\infty$, with a scale factor evolving as $a(\eta) = a_i + a_{\text{inf}}(\eta/\eta_{\text{inf}})^\beta$. Provided $d \ln \omega_{\text{phys}} / d \ln k_{\text{phys}}$ is not singular, the WKB approximation is valid asymptotically. One could reproduce the above calculation with the substitution for the new scale factor, and the final expression for the power spectrum would be the same [at late times, a_i becomes negligible compared to the expansion term in $a(\eta)$]. However the ratio η_2/η_1 then depends on k if η_2 is in the far past when the term a_i cannot be neglected even if $\beta = -1$.

We have derived the stress-energy tensor for a free scalar field with a general modified dispersion relation. In particular, we have obtained the expression of the energy density and pressure in a FLRW background. This result generalises previous studies [16,17] which were restricted to the Jacobson-Corley dispersion relation. We have applied our calculation of the stress-energy tensor to a series of examples to discuss the equation of state of trans-Planckian modes and the issue of backreaction in inflationary cosmology.

We have first examined in details the possibility that the energy contained in trans-Planckian modes with a frequency much smaller than the Hubble expansion rate could account for the form of vacuum energy density measured in the low-redshift Universe [18]. In the case of dispersion relations with ultralow frequencies at high momenta, as proposed in Ref. [18], one cannot select a vacuum initial state without ambiguity. Nevertheless, using well-motivated proposals for this initial state, we have shown that the equation of state of these trans-Planckian modes does not have the correct form, contrary to the claim made in Ref. [18]. Moreover, the numerical value of the energy density is not of the order of the critical energy density unless fine-tuning is required, and we thus conclude that the scenario proposed in Ref. [18] does not stand up to scrutiny.

We have also discussed the issue of backreaction in trans-Planckian inflationary cosmology. In particular, we have focused on a class of dispersion relations for which the pulsation $\omega_{\text{phys}} < H$ for a finite time interval for trans-Planckian comoving momenta. This is the general class of dispersion relations for which the WKB approximation is broken at some point during inflation, but is restored in the far past as $\eta \rightarrow -\infty$. If the evolution is adiabatic all throughout inflation up to horizon exit, it is known that the power spectrum is not modified. In the above case, the WKB approximation is precisely broken in the region where $\omega_{\text{phys}} < H$, but is valid at earlier and at later times. We have obtained the analytical expression for the amount of energy density stored in modes at late time, and find that it is in general much larger than the background energy density. We have computed the power spectrum of metric fluctuations and showed that this power spectrum is not modified, independently of the dispersion relation and whether WKB holds or not, if the inflationary period is strictly de Sitter. This supports the belief that inflation is robust to a change in the dispersion relation. If however the inflationary period is not strictly de Sitter, then the power spectrum is tilted with respect to the standard case (unmodified dispersion relation) and superimposed oscillations appear. Finally we have exhibited a class of dispersion relations for which the power spectrum of metric fluctuations is strongly modified and the initial conditions can be set up properly, since they are fixed in a region where the WKB approximation holds.

Our work thus completes the arguments developed in previous works [2,3,7,12] on the relation between adiabaticity of the mode evolution, the modification of the predictions of inflation, and the magnitude of backreaction. In particular the following picture seems to emerge: if the evolution of the modes is adiabatic all throughout inflation up to horizon exit, then the power spectrum is unmodified (or weakly modified), and backreaction is not an issue. If however adiabaticity is broken at some point, then modifications to the power spectrum are likely to appear (except if the background spacetime is very close to de Sitter), but effects of backreaction then appear generic, and one must use a higher order expansion of the Einstein equations to derive meaningful conclusions. Finally, there exists dispersion relations such that the backreaction is weak but modifications to the power spectrum are not negligible, notably when the ratio k_c/H_{inf} is not too large, when the time interval in which adiabaticity is broken is small, and when the background spacetime is not de Sitter. However this obviously requires fine-tuning, and overall a scale invariant power spectrum does indeed appear robust against changes in the dispersion relation if inflation takes place.

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APPENDIX A: NOTATIONS

This appendix provides some identities satisfied by the vector field u_μ , the projector $\perp_{\mu\nu}$ and the derivative \mathcal{D}_α that are implicitly used in the calculations of Section II. Since the norm of u_μ is conserved, one has trivially $u_\alpha \nabla_\mu u^\alpha = 0$. The covariant derivative of u_μ can be conveniently decomposed as

$$\nabla_\mu u_\nu = \frac{1}{3}\theta \perp_{\mu\nu} + \sigma_{(\mu\nu)} + \omega_{[\mu\nu]} - a_\mu u_\nu, \quad (\text{A1})$$

where $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$ and $A_{[\mu\nu]} \equiv (A_{\mu\nu} - A_{\nu\mu})/2$ are the symmetrised and antisymmetrised parts of $A_{\mu\nu}$ respectively. The shear $\sigma_{\mu\nu}$ and vorticity $\omega_{[\mu\nu]}$ are tracefree and purely spatial, i.e.

$$\sigma_{\mu\nu} g^{\mu\nu} = 0, \quad \sigma_{\mu\nu} u^\mu = 0, \quad \omega_{\mu\nu} g^{\mu\nu} = 0, \quad \omega_{\mu\nu} u^\mu = 0, \quad (\text{A2})$$

and $a_\mu \equiv u^\nu \nabla_\nu u_\mu$ is the acceleration of the observer comoving with u_μ . The quantity a_μ is spatial, i.e. $a_\mu u^\mu = 0$ and one has $a_\mu = 0$ for a geodesic. Using these relations, the expansion rate θ can be written as

$$\theta = \perp^{\mu\nu} \nabla_\mu u_\nu. \quad (\text{A3})$$

The integrability condition of the hypersurface Σ implies that $\omega_{\mu\nu} = 0$ which is indeed satisfied if the definition of u^μ given in the main text is imposed. The projection operator $\perp_{\mu\nu}$ verifies

$$\perp_\mu^\mu = 3, \quad \perp_\mu^\nu u^\mu = 0, \quad \perp_\alpha^\beta \perp_\beta^\gamma = \perp_\alpha^\gamma. \quad (\text{A4})$$

For the FLRW case, it follows that $\theta = 3H$, $a_\mu = \sigma_{\mu\nu} = \omega_{\mu\nu} = 0$.

Finally, if we denote by σ^i the internal coordinates on the hypersurface Σ defined by the embedding $x^\mu = \bar{x}^\mu(\sigma^i)$, then the three dimensional spatial metric is given by $\perp_{ij} = g_{\mu\nu} \partial_i \bar{x}^\mu \partial_j \bar{x}^\nu$ and it follows that $\mathcal{D}^2 \phi$ is given directly by

$$\mathcal{D}^2 \phi = \frac{1}{\sqrt{\perp}} \partial_i (\sqrt{\perp} \partial^i \phi), \quad (\text{A5})$$

where \perp is the determinant of \perp_{ij} . Hence $\mathcal{D}_\mu \mathcal{D}^\mu \phi$ is the three dimensional Laplacian as defined by the observer comoving with u^μ . With the decomposition Eq. (A1), it is easily shown using the identity $u^\alpha u^\beta \nabla_\alpha \nabla_\beta \phi = \ddot{\phi} - a^\alpha \nabla_\alpha \phi$ that

$$\mathcal{D}^2 \phi = \square \phi + \ddot{\phi} - a^\alpha \nabla_\alpha \phi + \theta \dot{\phi}, \quad (\text{A6})$$

where $\square \equiv \nabla_\mu \nabla^\mu$ denotes the four dimensional d'Alembertian.

APPENDIX B: DERIVATION OF THE STRESS-ENERGY TENSOR IN THE PARTICULAR CASE OF THE CORLEY-JACOBSON DISPERSION RELATION

We detail in this appendix the simple case where only b_{11} does not vanish in the corrective Lagrangian. This Lagrangian contains derivatives of the metric that must be varied to obtain the stress-energy tensor. With $\delta \sqrt{-g} / \delta g^{\mu\nu} = -\sqrt{-g} g_{\mu\nu} / 2$, this stress-energy tensor can be written as

$$T^{\mu\nu} = -2g_\alpha^\mu g_\beta^\nu \frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}} + \mathcal{L} g^{\mu\nu} - \frac{2}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} \frac{\delta \mathcal{L}}{\delta \partial_\rho g_{\mu\nu}} \right). \quad (\text{B1})$$

For the present case $\mathcal{L}_{\text{cor}} = -b_{11} (\mathcal{D}^2 \phi)^2$, it leads to

$$\begin{aligned} T^{\mu\nu} = & \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} (\partial_\alpha \phi \partial^\alpha \phi) g^{\mu\nu} + 2\lambda u^\mu u^\nu - b_{11} (\mathcal{D}^2 \phi)^2 g^{\mu\nu} \\ & + b_{11} \left[2g_\alpha^\mu g_\beta^\nu \frac{\delta (\mathcal{D}^2 \phi)^2}{\delta g^{\alpha\beta}} + \frac{2}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} \frac{\delta (\mathcal{D}^2 \phi)^2}{\delta \partial_\rho g_{\mu\nu}} \right) \right]. \end{aligned} \quad (\text{B2})$$

The first term in brackets can be written, using Eq. (9),

$$\frac{\delta (\mathcal{D}^2 \phi)}{\delta g^{\mu\nu}} = \nabla_{(\mu} \nabla_{\nu)} \phi + \dot{\phi} \nabla_{(\mu} u_{\nu)} + \theta u_{(\mu} \nabla_{\nu)} \phi + 2u^\alpha u_{(\mu} \nabla_{\nu)} \nabla_\alpha \phi - \perp^{\alpha\beta} \Gamma_{\alpha\beta(\mu} \nabla_{\nu)} \phi - \dot{\phi} g^{\alpha\beta} \Gamma_{\alpha\beta(\mu} u_{\nu)}, \quad (\text{B3})$$

while the second term gives

$$\frac{\delta (\mathcal{D}^2 \phi)}{\delta \partial_\rho g_{\mu\nu}} = -\frac{1}{2} \left[2 \perp^{\rho(\mu} \nabla^{\nu)} \phi - \perp^{\mu\nu} \nabla^\rho \phi + 2\dot{\phi} g^{\rho(\mu} u^{\nu)} - \dot{\phi} g^{\mu\nu} u^\rho \right]. \quad (\text{B4})$$

The derivative of this equation finally reads

$$\begin{aligned} \frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} \frac{\delta (\mathcal{D}^2 \phi)}{\delta \partial_\rho g_{\mu\nu}} \right) = & -\frac{1}{2} \nabla_\rho \left[\perp^{\rho(\mu} \nabla^{\nu)} \phi - \perp^{\mu\nu} \nabla^\rho \phi + \dot{\phi} g^{\rho(\mu} u^{\nu)} - \dot{\phi} g^{\mu\nu} u^\rho \right] \\ & + \perp^{\rho\alpha} \Gamma_{\rho\alpha}^{(\mu} \nabla^{\nu)} \phi + \dot{\phi} g^{\rho\alpha} \Gamma_{\rho\alpha}^{(\mu} u^{\nu)}. \end{aligned} \quad (\text{B5})$$

It can be checked that the terms involving Christoffel symbols in Eqs. (B3) and (B4) strictly vanish, a necessary condition to the covariance of the stress-energy tensor. Finally, we end up with

$$\begin{aligned} T_{\mu\nu} = & \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial_\alpha \phi \partial^\alpha \phi) g_{\mu\nu} + 2\lambda u_\mu u_\nu + T_{\mu\nu}^{(\text{F})} \\ & + b_{11} (\mathcal{D}^2 \phi) \left[2\square \phi \perp_{\mu\nu} + 2(\ddot{\phi} + \theta \dot{\phi}) g_{\mu\nu} - (\mathcal{D}^2 \phi) g_{\mu\nu} \right. \\ & \left. + 4u_{(\mu} \nabla^\alpha u_{\nu)} \nabla_\alpha \phi - 4u_{(\mu} \nabla_{\nu)} u^\alpha \nabla_\alpha \phi - 4a_{(\mu} \nabla_{\nu)} \phi \right] \\ & - 2b_{11} \left[2\nabla_{(\mu} \phi \nabla_{\nu)} (\mathcal{D}^2 \phi) + 2(\mathcal{D}^2 \phi)^\cdot u_{(\mu} \nabla_{\nu)} \phi - \perp_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha (\mathcal{D}^2 \phi) \right. \\ & \left. + 2\dot{\phi} u_{(\mu} \nabla_{\nu)} (\mathcal{D}^2 \phi) - \dot{\phi} (\mathcal{D}^2 \phi)^\cdot g_{\mu\nu} \right]. \end{aligned} \quad (\text{B6})$$

The expression of the Lagrange multiplier λ results from the equation of motion for u^μ

$$u^\mu \frac{\delta \mathcal{L}}{\delta u^\mu} = \frac{1}{\sqrt{-g}} u^\mu \partial_\nu \left(\sqrt{-g} \frac{\delta \mathcal{L}}{\delta \partial_\nu u_\mu} \right), \quad (\text{B7})$$

which gives

$$\lambda = b_{11} \left[(\mathcal{D}^2 \phi) \left(\ddot{\phi} + \theta \dot{\phi} - 2a^\alpha \nabla_\alpha \phi \right) - (\mathcal{D}^2 \phi)^\cdot \dot{\phi} \right] + \lambda^{(\text{F})}. \quad (\text{B8})$$

It follows that the energy density and the pressure measured by the observer comoving with u^μ (i.e. $\rho = T_{\mu\nu} u^\mu u^\nu$ and $p = T_{\mu\nu} \perp^{\mu\nu} / 3$) are respectively given by

$$\rho = \dot{\phi}^2 + \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + b_{11} (\mathcal{D}^2 \phi)^2 + u^\mu u^\nu T_{\mu\nu}^{(\text{F})} + 2\lambda^{(\text{F})}, \quad (\text{B9})$$

$$p = \frac{1}{3} \dot{\phi}^2 - \frac{1}{6} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{b_{11}}{3} (\mathcal{D}^2 \phi) [3 (\mathcal{D}^2 \phi) + 2a^\alpha \nabla_\alpha \phi] + \frac{2b_{11}}{3} \dot{\phi} (\mathcal{D}^2 \phi)^\cdot + \frac{2b_{11}}{3} \nabla_\alpha \phi \nabla^\alpha (\mathcal{D}^2 \phi) - \frac{1}{3} \perp^{\mu\nu} T_{\mu\nu}^{(\text{F})}. \quad (\text{B10})$$

In the case of the Minkowski metric, the mean values of the previous expressions calculated in a thermal state reduce to the formulas obtained in Refs. [16,17].

Finally, the field equation for ϕ is obtained by varying the action to obtain

$$\frac{\delta \mathcal{L} \sqrt{-g}}{\delta \phi} = \partial_\mu \left(\frac{\delta \mathcal{L} \sqrt{-g}}{\delta \partial_\mu \phi} \right) - \partial_{\mu\nu} \left(\frac{\delta \mathcal{L} \sqrt{-g}}{\delta \partial_{\mu\nu} \phi} \right). \quad (\text{B11})$$

hence

$$\square \phi = -2b_{11} [\mathcal{D}^2 (\mathcal{D}^2 \phi) + 2a^\nu \nabla_\nu (\mathcal{D}^2 \phi) + \nabla_\mu a^\mu (\mathcal{D}^2 \phi)]. \quad (\text{B12})$$

It can be checked that the general formulae given Section II, when applied to the special case of the present Corley-Jacobson dispersion relation, reduce to the equations given in this Appendix.